Math 245C Lecture 9 Notes

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1 The Schwarz Space

1.1 Topology of the Schwarz space

Definition 1.1. Given $N \geq 0$ and $\alpha \in \mathbb{N}^n$ ($\mathbb{N} = \{0, 1, 2, \dots\}$), we define the seminorm of $f \in C^\infty(\mathbb{R}^n)$

$$\|f\|_{(N,\alpha)} := \sup_x (1 + |x|)^N |\partial^\alpha f(x)|.$$

The **Schwarz space** is $\mathcal{S} = \{f \in C^\infty(\mathbb{R}^n) : \|f\|_{(N,\alpha)} < \infty \, \forall N \in \mathbb{N}, \alpha \in \mathbb{N}^n\}$.

Example 1.1. If $f \in C^\infty(\mathbb{R}^n)$ with compact support, then $f \in \mathcal{S}$.

Example 1.2. $|\partial^\alpha(e^{-|x|^2})| \leq c(1 + |x|^{2|\alpha|})e^{-|x|^2}$.

$\mathcal{S}$ is endowed with a topology induced by the seminorm as follows: $(f_k)_k \subseteq \mathcal{S}$ converges to $f \in \mathcal{S}$ iff

$$\lim_{k \to \infty} \|f_k - f\|_{(N,\alpha)} = 0$$

for all $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$. Recall that a Freéchet is a complete, Hausdorff, topological vector space whose topology is induced by a countable family of seminorms.

Proposition 1.1. *$\mathcal{S}$ is a Fréchet space.*

Proof. Hausdorff: Given $f \in \mathcal{S}$ and $\varepsilon > 0$, $U^\varepsilon_{(N,\alpha)} = \{g \in \mathcal{S} : \|f - g\|_{(N,\alpha)} < \varepsilon\}$ are the open sets that generate the topology of $\mathcal{S}$. Let $f_1, f_2 \in \mathcal{S}$ be distinct. Let $x_0 \in \mathbb{R}^n$ be such that $4\delta := |f_1(x_0) - f_2(x_0)| > 0$. Since $|f_1 - f_2|$ is continuous, there exists an open neighborhood O of x_0 such that $|f_1(x) - f_2(x)| \geq 3\delta$ for all $x \in O$. We have $U^\delta_{(0,0)}(f_1) \cap U^\delta_{(0,0)}(f_2) = \varnothing$. This proves that $\mathcal{S}$ is a Hausdorff space.

Completeness: Let $(f_k)_k \subseteq \mathcal{S}$ be a Cauchy sequence: $\lim_{k,\ell \to \infty} \|f_k - f_\ell\|_{(N,\alpha)} = 0$ for all $N \in \mathbb{N}, \alpha \in \mathbb{N}^n$. Taking $N = 0$ for each α, we obtain that $(\partial^\alpha f_k)_k$ is a Cauchy sequence for the uniform norm, and so $(\partial^\alpha f_k)_k$ converges uniformly to some $g_\alpha \in C(\mathbb{R}^n)$. We claim that $\sup_x (1+|x|)^N g_\alpha(x) < \infty$. We have $(1|x|)^n |\partial^\alpha f_k - \partial^\alpha f_\ell| \le \varepsilon$ for large k, ℓ. Letting $\ell \to \infty$, we get $1|x|)^n |\partial^\alpha f_k - g_\alpha|$ for large k. Then

$$(1+|x|^N |g_\alpha| \le \underbrace{(1+|x|)^N |g_\alpha(x) - \partial^\alpha f_k(x)|}_{\le \varepsilon} + (1+|x|)^N |\partial^\alpha f_k(x)| < \infty.$$

It remains to show that $g_0 \in C^\infty(\mathbb{R}^n)$ and $\partial^\alpha g_0 = g_\alpha$. By Taylor's expansion,

$$f_k(x+h) = f_k(x) - \nabla f_k(x) h = \int_0^1 \int_0^1 (\nabla^2 f_k(x+tsh))h \cdot h)\, ds\, dt.$$

Thus,

$$|f_k(x+h) - f_k(x) - h \cdot \nabla f_j(x)| \le \frac{|h|^2}{2} M, \qquad M = \sup_k \sup_{|\alpha|=2} \|f_k\|_{(0,\infty)}.$$

Letting $k \to \infty$, we obtain

$$\left| g_0(x+h) - g_0(x) - \sum_{i=1}^n g_{(0,\cdots,0,1,0,\dots,0)}(x) h_i \right| \le \frac{M}{2} \|h\|^2.$$

Since $g_{(0,\cdots,0,1,0,\dots,0)}(x)$ is continuous, we conclude that g_0 is differentiable at x and that $\frac{\partial}{\partial x_i} g_0(x) = g_{(0,\dots,0,1,0,\dots,0)}(x)$. Increasing the rank of the expansion, we obtain the desired result. So $g_\alpha = \partial^\alpha f$. □

1.2 Equivalent characterizations of functions in the Schwarz space

Proposition 1.2. *Let $f \in C^\infty(\mathbb{R}^n)$. The following are equivalent:*

1. *$f \in \mathcal{S}$.*
2. *$x^\beta \partial^\alpha f$ is bounded for any $\beta, \alpha \in \mathbb{N}^n$.*
3. *$\partial^\alpha (x^\beta f)$ is bounded for any $\beta, \alpha \in \mathbb{N}^n$.*

Proof. (1) $\implies$ (2): Let $\alpha, \beta \in \mathbb{N}^n$. Then

$$|x^\beta| |\partial^\alpha f(x)| \le (1+|x|)^{|\beta|} |\partial^\alpha f(x)| \le \|f\|_{(|\beta|,\alpha)}.$$

(2) $\implies$ (3): We have

$$\partial^\alpha (x^\beta f) = \sum_{a \in A, b \in B} x^a \partial^b f,$$

where A and B are finite sets determined by α, β. Thus,

$$|\partial^\alpha (x^\beta f(x))| \le \sum_{a \in A, b \in B} \|x^\alpha \partial^b \beta\| < \infty.$$

(3) $\implies$ (1): We have $\|\partial^\alpha f\|_\infty < \infty$ for all $\alpha \in \mathbb{N}^n$. It remains to show that $\|(1+|x|)^N \partial^\alpha f(x)\|_\infty < \infty$. Fix an integer $N \geq 1$. Then

$$\delta_N := \min\{\sum_{i=1}^n |x_i|^N : \|x\| = 1\} > 0.$$

Hence,

$$\delta_N \leq \sum_{i=1}^n \left|\frac{x_i}{\|x\|}\right|^N = \frac{1}{\|x\|^N} \sum_{i=1}^N |x_i|^N.$$

So

$$\|x\|^N \leq \frac{1}{\delta_N} \sum_{i=1}^n |x_i|^N.$$

It remains to show that $\||x_i|^N \partial^\alpha f\|_\infty < \infty$. We have for $N = 1$ that

$$\partial_{x_j}(x_i \partial^\alpha f) = \delta_{i,j} \partial^\alpha f + x_i \partial_{x_j} \partial^\alpha f,$$

so

$$\|x_i \partial_{x_j} \partial^\alpha f\| \leq \|\partial_{x_j}(x_i \partial^\alpha f)\|_\infty + \|\partial \alpha^\alpha f\|_\infty^j.$$

Repeat the process for $N = 2, 3, \ldots$. $\square$